On the multichannel Kondo model

1994 J. Phys.: Condens. Matter 6 4547

(http://iopscience.iop.org/0953-8984/6/24/016)

J. Phys.: Condens. Matter 6 (1994) 4547–4568. Printed in the UK

On the multichannel Kondo model

Junwu Gan

Department of Physics, The University of British Columbia, 6224 Agricultural Road, Vancouver, BC V6T 1Z1, Canada

Received 7 October 1993, in final form 8 February 1994

Abstract. A detailed and comprehensive study of the one-impurity multichannel Kondo model is presented. In the limit of a large number of conduction electron channels $k \gg 1$, the low-energy fixed point is accessible to a renormalization group improved perturbative expansion in $1/k$. This straightforward approach enables us to examine the scaling, thermodynamics and dynamical response functions in great detail and make clear the following features: (i) the criticality of the fixed point, (ii) the universal non-integer degeneracy, and (iii) that the compensating spin cloud has a spatial extent of the order of one lattice spacing.

1. Introduction

Recently, the non-Fermi-liquid infrared fixed point of the multichannel Kondo model, which describes a system of k identical conduction bands interacting with an impurity spin S, has received considerable attention [1–18]. The interest is aroused by its potential application in the metallic glasses [1, 5, 13], and heavy-fermion uranium alloys [2–4] where more examples have been found with their low-temperature behaviour falling outside the usual Fermi liquid expectation [19, 20]. It is further enhanced by the more interesting observation of non-Fermi-liquid behaviour in the normal state of the high-T_c cuprates [21, 22] and its resemblance to the low-energy behaviour of the multichannel Kondo model [6, 15].

Although this model is more than ten years old [23] and has been studied by various methods [9, 14, 15, 17, 24–26], there is still room left for a simple interpretation of the physics of the low-energy fixed point, which is often said to be non-trivial. This task is easily achieved in the limit $k \gg 1$, where the low-energy fixed point has a value of order $1/k$ for the coupling constant and is accessible to a renormalization group (RG) improved perturbative expansion in $1/k$. Some physical quantities difficult to calculate by the other methods are readily obtained by this perturbative approach and the results provide new insight into the fixed point. That the nature of the fixed point remains qualitatively the same when continuing k to small values as long as $k > 2S$ has been demonstrated by the results of the Bethe *ansatz* or the conformal field method [9, 25, 26]. In this paper, we calculate a long list of physical quantities to leading or sub-leading order in $1/k$. A simple physical picture can be sketched based on these results and the previous understanding.

The underlying physics of the overscreened Kondo problem has been explained by Nozières and Blandin [23]. For an antiferromagnetic Kondo interaction between the impurity spin and the conduction electrons, the impurity spin pulls in conduction electrons with opposite spin. This increases the spin-flip exchange which in turn enhances the attraction of the conduction electrons with opposite spins toward the impurity spin. The result of this cooperative enhancement of the Kondo interaction tends to pull in one conduction electron from each of the k channels to screen the impurity spin. However, unlike the case

0953-8984/94/244547+22$19.50

$k = 2S$, the ground state with an infinitely strong effective Kondo coupling is unstable. Since $k > 2S$, the conduction electrons would overscreen the impurity spin, resulting in a composite object composed of k spin-aligned electrons antiferromagnetically bound with the impurity spin. As shown by Nozières and Blandin [23], the residual interaction between this composite object with a net spin $k/2 - S$ and the conduction electrons is antiferromagnetic, and therefore will grow under renormalization. Additional conduction electrons must come in to screen this composite object. The process keeps going on, resulting in a critical system. It is important to realize that the system is only critical along the time axis. As short-time details are averaged out, the system approaches a universal limiting behaviour governed by the infrared fixed point. The impurity spin is asymptotically screened in the ground state, which is a spin singlet. However, the infrared fixed point can be reached only in an infinite system in which the energy levels form a continuum. In finite systems, the typical distance between discrete energy levels, of order $1/L$ for a system of linear dimension L, cuts off the scaling toward the fixed point. The ground state of a finite system has a residual spin and is doubly degenerate. A common measure of the residual spin is $\sqrt{T\chi(T)}$ as the temperature $T \to 0$, where $\chi(T)$ is the magnetic susceptibility. As we shall see, it is non-zero for a finite system. We shall also show that the entropy as given by the coefficient of the linear T term in the free energy is indeed $\ln(2S + 1)$ for a finite system.

An intriguing feature of the multichannel Kondo model is that the entropy of an infinite system reduces to a smaller universal value [10, 15, 26] although the ground state is a spin singlet. If the entropy is still defined as the logarithm of the ground-state degeneracy, the ground state could have a universal non-integer degeneracy. Since the entropy is deduced by calculating the free energy at a small but finite temperature T which cuts off the scaling toward the fixed point, it is natural to relate the entropy to the effective residual spin at T. A necessary implication of this relation is that a tiny magnetic field will lift the degeneracy at $T \to 0$, as we shall see later.

Another interesting issue pertaining to all kinds of Kondo problem, either overscreened or exactly screened, is the spatial size of the conduction electron screening cloud. This is a point underlying the resonant level nature of the Kondo problems. Considering the one-channel Kondo problem, it is well known that below the Kondo temperature T_K the effective Kondo interaction enters the strong-coupling regime. The energy gain from screening the impurity spin is of order T_K. If the screening cloud formed a localized bound state with the impurity spin, it would have a spatial spreading of $v_F/T_K(\gg 1/k_F)$ [27]. NMR experiments ruled out any conduction electron screening cloud bigger than one lattice spacing [28]. Although the physical argument for the screening cloud to have a size of $1/k_F$ has been given [29, 30], here we calculate the Knight shift for the multichannel Kondo model and explicitly show that the only length scale is $1/k_F$. We also explain why the same conclusion can be extended to the exactly screened case of $k = 1$.

The paper is organized as follows. In section 2 the multichannel Hamiltonian and the Popov method are briefly recalled. In section 3, the general integral expression for the conduction electron self-energy to order $O(k^{-4})$ is derived, which can serve as a future reference. In section 4, the integrals in the self-energy are evaluated at $T = 0$ and the results are used to derive the RG equation and the running coupling constant. In section 5, the scaling solutions for the conduction electron scattering rate and resistivity are obtained. The free energy is calculated in section 6, from which the specific heat and entropy are deduced. The magnetic susceptibility and field-dependent magnetization are calculated in section 7 for an equal-spin gyromagnetic ratio for the conduction electron and impurity spin. The dynamical susceptibility for the impurity spin is calculated in section 8 and the relaxation rate is deduced. The general case with different gyromagnetic ratios is considered in section 9. The Knight shift in the space surrounding the impurity spin is calculated in section 10. The last section is devoted to a discussion of related issues. Some of the results have been briefly reported in [15]. Some details for the dynamical spin correlation functions are included in appendix B.

2. Popov technique

Without losing generality, we consider the local impurity to be a spin $S = \frac{1}{2}$. The multichannel Kondo Hamiltonian in the magnetic field is

$$H = \sum_{\substack{k,\mu=\pm \\ \lambda=1...k}} (\epsilon_k + \mu h) c^\dagger_{k\mu\lambda} c_{k\mu\lambda} + 2\mu_S \boldsymbol{h} \cdot \boldsymbol{S} + \frac{J}{\mathcal{N}} \sum_{k,k'} \sum_{\substack{\mu,\nu=\pm \\ \lambda=1...k}} c^\dagger_{k\mu\lambda} \frac{\boldsymbol{\sigma}_{\mu\nu}}{2} c_{k'\nu\lambda} \cdot \boldsymbol{S} \tag{1}$$

where $\boldsymbol{S}$ is a spin-1/2 operator, $\boldsymbol{\sigma}_{\mu\nu}$ are the Pauli matrices, J is the Kondo interaction strength and $\mathcal{N}$ is the number of lattice sites. We have set the conduction electron gyromagnetic ratio and Bohr magneton equal to one so that the magnetic field h has dimensions of energy. We have introduced a parameter μ_S to account for possible differences between the conduction electron and impurity spin: μ_S is equal to the gyromagnetic ratio of the impurity spin divided by that of the conduction electron. We shall adopt the usual cut-off scheme for the conduction electron band: $-D < \epsilon_k < D$, with a constant density of states ρ per spin per channel.

Representing the impurity spin in terms of pseudofermions

$$\boldsymbol{S} = \frac{1}{2} \sum_{\mu,\nu=\pm} f^\dagger_\mu \boldsymbol{\sigma}_{\mu\nu} f_\nu$$

Popov made the observation that the Kondo Hamiltonian without imposing constraints on the pseudofermions is disconnected in the pseudofermion charge sectors, i.e.

$$\mathcal{Z} = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \mathcal{Z}_0 + \mathcal{Z}_1 + \mathcal{Z}_2 = \mathrm{Tr}\, \delta(\hat{n}_f) \mathrm{e}^{-\beta H} + \mathrm{Tr}\, \delta(\hat{n}_f - 1) \mathrm{e}^{-\beta H} + \mathrm{Tr}\, \delta(\hat{n}_f - 2) \mathrm{e}^{-\beta H}.$$

Among these three separate contributions, only $\mathcal{Z}_1$ from the subspace

$$\hat{n}_f = \sum_{\sigma=\pm} f^\dagger_\sigma f_\sigma = 1$$

is physical. Popov's technique [31] is to add an imaginary chemical potential, $\mathrm{i}\omega_0 = \mathrm{i}\pi/(2\beta)$, to the pseudofermions so that $\mathcal{Z}_0 + \mathcal{Z}_2 = 0$. Using this trick, the partition function in the path integral formalism can be represented as

$$\mathcal{Z} = \int \mathcal{D}[\bar{c}, c, \bar{f}, f] \exp\left(- \int_0^\beta \mathrm{d}\tau (\mathcal{L}_0 + H) \right) \tag{2}$$

$$\mathcal{L}_0 = \sum_{k,\mu,\lambda} \bar{c}_{k\mu\lambda} \partial_\tau c_{k\mu\lambda} + \sum_\mu \bar{f}_\mu (\partial_\tau + \mathrm{i}\omega_0) f_\mu. \tag{3}$$

There is no additional constraint. The standard perturbation method then follows from this path integral representation. The impurities are assumed to be randomly distributed in space. The averaging over the impurity distribution is done according to the standard recipe, as in the case of the spinless impurities [32, 33], and we only keep contributions linear in the impurity density n_{i}. The Feynman rules for constructing diagrams are listed in appendix A.

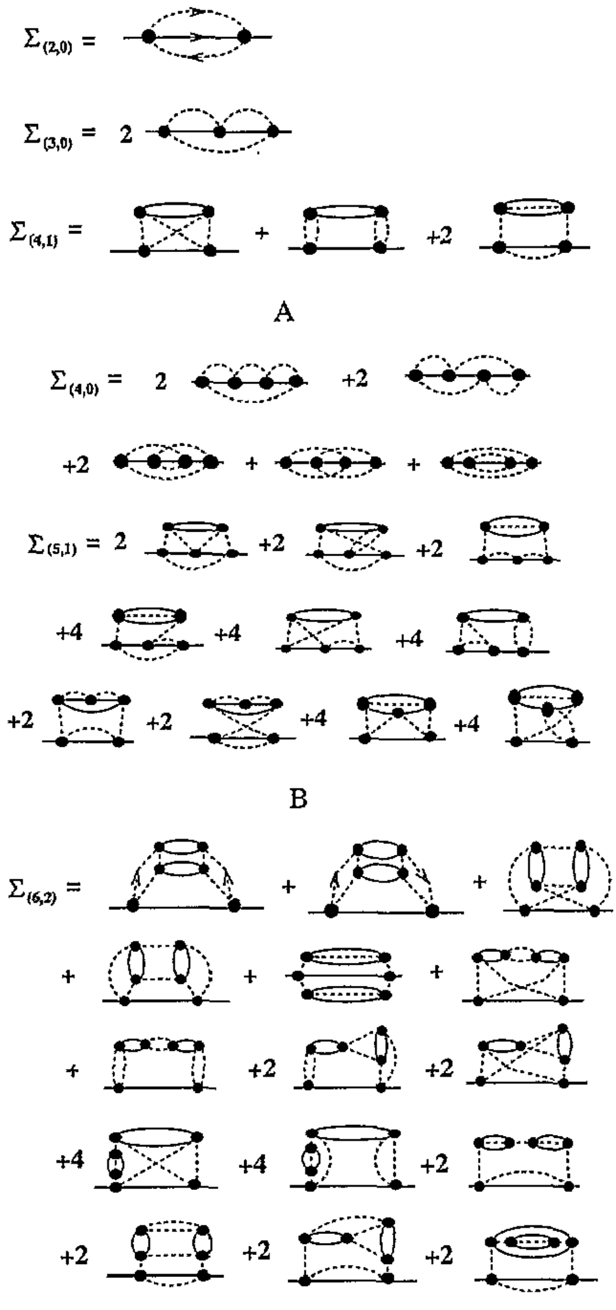


Figure 1. Feynman diagrams for the conduction electron self-energy. The arrows on the propagators are marked only when they make a difference.

3. Conduction electron self-energy

The first thing we shall calculate is the conduction electron self-energy in the absence of the external magnetic field. Up to order J^4, J^5k and J^6k^2, the relevant diagrams are given in figure 1. The two subscripts in each term of the self-energy expansion indicate the powers of J and k respectively. Since each conduction electron loop must contain at least two interacting vertices, one can immediately convince oneself that all contributions up to order k^{-4} have been included in figure 1 when $J\rho$ is counted as $1/k$. After lengthy algebra, the final results at finite temperature are

$$\Sigma(\mathrm{i}\omega_n, T) = \Sigma_{(2,0)} + \Sigma_{(3,0)} + \Sigma_{(4,1)} + \Sigma_{(4,0)} + \Sigma_{(5,1)} + \Sigma_{(6,2)} \tag{4}$$

$$\Sigma_{(2,0)}(\mathrm{i}\omega_n) = -\frac{3n_\mathrm{i}}{16}J^2\rho\int\frac{\mathrm{d}\epsilon}{\mathrm{i}\omega_n - \epsilon} \tag{5}$$

$$\Sigma_{(3,0)}(\mathrm{i}\omega_n) = \frac{3n_\mathrm{i}}{16}J^3\rho^2\int \mathrm{d}\epsilon\,\mathrm{d}\epsilon_1\frac{\tanh(\frac{1}{2}\beta\epsilon_1)}{(\mathrm{i}\omega_n - \epsilon)(\epsilon - \epsilon_1)} \tag{6}$$

$$\Sigma_{(4,1)}(\mathrm{i}\omega_n) = \frac{3n_\mathrm{i}}{16}J^4\rho^3k\int\frac{\mathrm{d}\epsilon\,\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\varphi^{\epsilon}_{\epsilon_1\epsilon_2}}{(\epsilon - \epsilon_1)(\mathrm{i}\omega_n - \epsilon_2)(\mathrm{i}\omega_n - \epsilon_2 + \epsilon - \epsilon_1)} \tag{7}$$

$$\Sigma_{(4,0)}(\mathrm{i}\omega_n) = \frac{9n_\mathrm{i}}{32}J^4\rho^3\int\frac{\mathrm{d}\epsilon\,\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2}{(\mathrm{i}\omega_n - \epsilon)(\epsilon - \epsilon_1)(\epsilon - \epsilon_2)}(\varphi^{\epsilon_1}_{\epsilon_2} - \tfrac{3}{8}) \tag{8}$$

$$\begin{aligned}\Sigma_{(5,1)}(\mathrm{i}\omega_n) = &-\frac{3n_\mathrm{i}}{32}J^5\rho^4k\int\frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\xi^{\epsilon_1}_{\epsilon_2\epsilon_3}}{(\epsilon_1 - \epsilon_2)(\epsilon_4 - \epsilon_3)(\epsilon_1 - \epsilon_2 + \epsilon_4 - \epsilon_3)(\mathrm{i}\omega_n - \epsilon_4)}\\ &- \frac{3n_\mathrm{i}}{32}J^5\rho^4k\int\frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\tanh(\frac{1}{2}\beta\epsilon_3)\,\varphi^{\epsilon_2}_{\epsilon_4\epsilon_1}}{(\epsilon_1 - \epsilon_2)(\epsilon_4 - \epsilon_3)(\epsilon_1 - \epsilon_2 + \epsilon_4 - \epsilon_3)(\mathrm{i}\omega_n - \epsilon_4 + \epsilon_2 - \epsilon_1)}\\ &+\frac{3n_\mathrm{i}}{16}J^5\rho^4k\int\frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\tanh(\frac{1}{2}\beta\epsilon_3)\,\varphi^{\epsilon_2}_{\epsilon_4\epsilon_1}}{(\epsilon_1 - \epsilon_2)(\mathrm{i}\omega_n - \epsilon_4)(\mathrm{i}\omega_n - \epsilon_4 + \epsilon_2 - \epsilon_1)}\left(\frac{1}{\epsilon_4 - \epsilon_3} + \frac{1}{\epsilon_2 - \epsilon_3}\right)\\ &- \frac{21n_i}{128}J^5\rho^4k\int\frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\tanh(\frac{1}{2}\beta\epsilon_3)\,\varphi^{\epsilon_2}_{\epsilon_4\epsilon_1}}{(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3)(\mathrm{i}\omega_n - \epsilon_4 + \epsilon_2 - \epsilon_1)}\end{aligned} \tag{9}$$

$$\begin{aligned}\Sigma_{(6,2)}(\mathrm{i}\omega_n) = &-\frac{3n_\mathrm{i}}{32}J^6\rho^5k^2\int\frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2\,\mathrm{d}\epsilon_3\,\mathrm{d}\epsilon_4\,\mathrm{d}\epsilon_5}{(\mathrm{i}\omega_n - \epsilon_5)(\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_4)}\\ &\times\left[\frac{\varphi^{\epsilon_1}_{\epsilon_2}\varphi^{\epsilon_3}_{\epsilon_4\epsilon_5}}{(\mathrm{i}\omega_n - \epsilon_5 + \epsilon_3 - \epsilon_4)(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)} + \frac{\varphi^{\epsilon_1\epsilon_3}_{\epsilon_2\epsilon_4\epsilon_5}}{\mathrm{i}\omega_n - \epsilon_5 + \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}\right.\\ &\left.\times\left(\frac{2}{\mathrm{i}\omega_n - \epsilon_5 + \epsilon_3 - \epsilon_4} + \frac{1}{\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4}\right)\right]\end{aligned} \tag{10}$$

where the integration range is always $[-D, D]$. Whenever necessary, the above integrals (and integrals throughout this paper) take the principal value. We have introduced the following shorthand notations:

$$\varphi^{\epsilon_1}_{\epsilon_2} = f(\epsilon_1)f(-\epsilon_2) + f(-\epsilon_1)f(\epsilon_2)$$

$$\varphi^{\epsilon_3}_{\epsilon_4\epsilon_5} = f(\epsilon_3)f(-\epsilon_4)f(-\epsilon_5) + f(-\epsilon_3)f(\epsilon_4)f(\epsilon_5)$$

$$\xi^{\epsilon_1}_{\epsilon_2\epsilon_3} = f(\epsilon_1)f(-\epsilon_2)f(-\epsilon_3) - f(-\epsilon_1)f(\epsilon_2)f(\epsilon_3)$$

$$\varphi^{\epsilon_1\epsilon_3}_{\epsilon_2\epsilon_4\epsilon_5} = f(\epsilon_1)f(-\epsilon_2)f(\epsilon_3)f(-\epsilon_4)f(-\epsilon_5) + f(-\epsilon_1)f(\epsilon_2)f(-\epsilon_3)f(\epsilon_4)f(\epsilon_5)$$

where $f(\epsilon) = 1/(\mathrm{e}^{\beta\epsilon} + 1)$ is the Fermi–Dirac function.

4. Renormalization group equation and solution

Under the analytic continuation $i\omega_n \to \omega + i0^+$, $\Sigma(\omega + i0^+) = \Sigma'(\omega) + i\Sigma''(\omega)$. The imaginary part of the self-energy $\Sigma''(\omega)$ is proportional to the total scattering rate of the conduction electron of energy ω. At $T = 0$, the integrations in the imaginary part of the self-energy (5)–(10) can be carried out and we obtain

$$\Sigma''(\omega, D, g) = \frac{3\pi n_i}{16\rho} g^2\big(P_0 + P_1 g \ln\widetilde{\omega} + P_2 g^2 k \ln\widetilde{\omega} + P_3 g^2 k + P_4 g^2 \ln^2\widetilde{\omega} + P_5 g^3 k \ln^2\widetilde{\omega} + P_6 g^3 k \ln\widetilde{\omega} + P_7 g^3 k + P_8 g^4 k^2 \ln^2\widetilde{\omega} + P_9 g^4 k^2 \ln\widetilde{\omega} + P_{10} g^4 k^2\big) \qquad (11)$$

where $g = J\rho$ and $\widetilde{\omega} = \omega/D$. Since we are only interested in the energy range $\omega \ll D$, we have neglected terms of powers of $\widetilde{\omega}$ in (11). All coefficients P_0 to P_{10} are known: $P_0 = 1$, $P_1 = -2$, $P_2 = 1$, $P_3 = \ln 2 - 1$, $P_4 = 3$, $P_5 = -\frac{7}{2}$, $P_6 = 5 - 3\ln 2$, $P_8 = 1$, $P_9 = 2\ln 2 - \frac{5}{2}$. The coefficients P_7 and P_{10} are not needed for deriving the RG equation to sub-leading order. However, they can be found from (9) and (10). Since we expect the fixed point g^* to be of order $1/k$ [23], our result (11) includes all contributions up to order $\mathcal{O}(k^{-4})$.

Since the dimensionless scattering rate deduced from (11) must be invariant under RG transformation, we obtain the following equation:

$$\left(\frac{\partial}{\partial \ln D} + \beta(g)\frac{\partial}{\partial g}\right)\rho\Sigma''(\omega, D, g) = 0. \qquad (12)$$

The beta function, $\beta(g)$, of the Kondo interaction g can be deduced from (11) using the standard RG technique [34]. Equation (12) can be regarded as an equation that generates logarithmic series with fixed g:

$$\rho\Sigma''(\omega, D, g) = -\beta(g)\frac{\partial}{\partial g}\int d(\ln D)\, \rho\Sigma''(\omega, D, g) + \text{constant}. \qquad (13)$$

Substituting (11) into (13), the integration over $\ln D$ can be carried out easily. Expressing $\beta(g)$ to sub-leading order as

$$\beta(g) = g^2(\kappa_1 + \kappa_2 g k + \kappa_3 g + \kappa_4 g^2 k + \kappa_5 g^3 k^2) \qquad (14)$$

and substituting it into (13), we obtain the following equations for the coefficients κ_1 to κ_5 from (13) by equating order by order the coefficients of the polynomials on the left- and right-hand sides:

$$\kappa_1 = \frac{P_1}{2P_0} = -1 \qquad (15)$$

$$\kappa_2 = \frac{P_2}{2P_0} = \frac{1}{2} \qquad (16)$$

$$\kappa_3 = 0 \qquad (17)$$

$$\kappa_4 = \frac{P_6 - 4P_3\kappa_1}{2P_0} = \frac{1 + \ln 2}{2} \qquad (18)$$

$$\kappa_5 = \frac{P_9 - 4P_3\kappa_2}{2P_0} = -\frac{1}{4}. \qquad (19)$$

There are three consistency equations

$$P_4 = \tfrac{3}{2}P_1\kappa_1 \qquad P_5 = 2P_2\kappa_1 + \tfrac{3}{2}P_1\kappa_2 \qquad P_8 = 2P_2\kappa_2$$

which are all satisfied.

The beta function obtained can be written explicitly as

$$\beta(g) = -g^2 + \tfrac{1}{2}kg^3 + \tfrac{1}{2}k(1+\ln 2)g^4 - \tfrac{1}{4}k^2g^5. \tag{20}$$

The intermediate-coupling fixed point g^*, determined by $\beta(g^*) = 0$, and the slope of the $\beta(g)$ at the fixed point, are then easily found to be

$$g^* = \frac{2}{k}\left(1 - \frac{2\ln 2}{k}\right) \tag{21}$$

$$\Delta = \beta'(g^*) = \frac{2}{k}\left(1 - \frac{2}{k}\right). \tag{22}$$

We find indeed $g^* \sim 1/k$, thus allowing a reliable expansion in the $k \to \infty$ limit. The slope Δ that determines the critical exponent is universal. The perturbative result (22) agrees with the conformal field result $2/(k+2)$ [9] up to sub-leading order in the $1/k$ expansion.

The running coupling constant $g_R(\omega)$ is determined by the differential equation

$$\frac{\mathrm{d}g_R}{\mathrm{d}\ln\omega} = \beta(g_R) \tag{23}$$

with the initial condition $g_R(\omega = D) = g$. With our perturbative $\beta(g)$ of (20), the solution of (23) covering the full range of energy scale from $\omega \ll T_K$ to $\omega \gg T_K$ can be obtained. We rewrite (23) in the integral form

$$\int_D^{\omega} \mathrm{d}\ln\omega' = \int_g^{g_R(\omega)} \frac{\mathrm{d}g'}{\beta(g')} \simeq g^* \int_g^{g_R(\omega)} \frac{\mathrm{d}g'}{(g')^2(g'-g^*)}\left[1 + g'\ln 2 - \tfrac{1}{2}k(g')^2\right]^{-1}.$$

This leads to the full solution

$$|g_R(\omega) - g^*| = |g - g^*|\left(\frac{\omega}{T_K}\right)^{\Delta}\left[g_R(\omega)\right]^{k\Delta/2}\mathrm{e}^{-\Delta/g_R(\omega)} \tag{24}$$

where $T_K = Dg^{k/2}\exp(-1/g)$. At $\omega < T_K$, it has an asymptotic form of

$$g_R(\omega) = g^* - \zeta\left(\frac{\omega}{T_K}\right)^{\Delta} \qquad \zeta = (g^* - g)(g^*)^{k\Delta/2}\mathrm{e}^{-\Delta/g^*}. \tag{25}$$

For an initial condition of weak coupling, $g \to 0$ and $D \to \infty$, the constant $\zeta = (g^*)^{1+k\Delta/2}\,\mathrm{e}^{-\Delta/g^*}$. From (24), it is interesting to note that the running coupling constant has a power law behaviour not only at low energy $\omega < T_K$ but also at high energy $\omega > T_K$, underlying the critical nature of the system.

5. Scattering rate and resistivity

To obtain the scaling solution for the conduction electron scattering rate $\rho\Sigma''$, we can use the RG invariance

$$\rho\Sigma''\big(\omega, D, g_R(D) = g\big) = \rho\Sigma''\big(\omega, D', g_R(D')\big). \tag{26}$$

Choosing $D' = \omega$, the logarithmic terms in $\rho\Sigma''\big(\omega, \omega, g_R(\omega)\big)$ of (11) drop out and the scaling form for (11) is

$$\rho\Sigma''(\omega, D, g) = \frac{3\pi n_i}{16}\left[g_R^2(\omega) - (1 - \ln 2)k g_R^4(\omega)\right] \simeq \frac{3\pi n_i}{4(k+2)^2}\left[1 - A_0\left(\frac{\omega}{T_K}\right)^{\Delta}\right] \tag{27}$$

where $A_0 = k\zeta[1 - (4 - 6\ln 2)/k]$. A similar RG procedure will be performed repeatedly later. Note that instead of a Lorentzian frequency dependence in the exactly screened Kondo problem, $\rho\Sigma'' \sim T_K^2/(\omega^2 + T_K^2)$, the scattering rate (27) has a cusp at $\omega = 0$. At non-zero temperatures, the cusp is expected to be smoothed out.

In order to calculate the resistivity, we need the temperature- and frequency-dependent conduction electron relaxation time. Identifying the transport relaxation time due to the Kondo exchange as $\tau_{ex}(\omega, T) = 1/[2\Sigma''(\omega, T)]$ since there is only s-wave scattering [32], the total scattering rate is $1/\tau(\omega, T) = 1/\tau_0 + 1/\tau_{ex}(\omega, T)$, where τ_0 is the ordinary relaxation time in the absence of the Kondo interaction. The ordinary scattering $1/\tau_0$ could arise from spinless impurities or defects which are assumed to be located at different lattice sites and uncorrelated with the impurity spins. A low impurity spin density is assumed, $n_i \ll 1$, such that $\tau_0 \ll \tau_{ex}$. The total relaxation time is substituted into the conductivity expression [33]

$$\sigma(T) = \frac{n_e e^2}{m_e}\int_0^\infty d\omega \frac{\tau(\omega)}{2T\cosh^2(\omega/2T)} \simeq \frac{n_e e^2}{m_e}\int_0^\infty d\omega \frac{\tau_0}{2T\cosh^2(\omega/2T)}\left(1 - \frac{\tau_0}{\tau_{ex}(\omega)}\right) \tag{28}$$

with e, m_e and n_e denoting the conduction electron charge, mass and density respectively. With our perturbative expression for $\Sigma''(\omega, T)$ to order k^{-3} from (5)–(7), we find for the resistivity due to Kondo scattering

$$\begin{aligned}\delta\rho_e(T) &= \frac{3\pi}{16}\frac{m_e}{n_e e^2 \rho} n_i g^2 \int_0^\infty \frac{dx}{\cosh^2\frac{1}{2}x}\left[1 - g\int_{-D/T}^{D/T} dy \frac{\tanh\frac{1}{2}y}{x - y} - g^2 k\Psi\left(x, \frac{D}{T}\right)\right] \\ &= \frac{3\pi}{8}\frac{m_e}{n_e e^2 \rho} n_i g^2\left(1 + O(g, g^2 k) + O(g, g^2 k) \times \ln\frac{D}{T}\right)\end{aligned} \tag{29}$$

where

$$\Psi\left(x, \frac{D}{T}\right) = \int_{-D/T}^{D/T}\int_{-D/T}^{D/T} \frac{dy\, dy_1}{(y - y_1)^2}\frac{1}{e^y + 1}\frac{1}{e^{-y_1} + 1}\left(1 - \frac{1}{e^{y_1 - y - x} + 1} - \frac{1}{e^{y_1 - y + x} + 1}\right). \tag{30}$$

Some difficult integrals have to be evaluated to obtain the sub-leading terms in (29). So we limit ourselves to the leading order. Performing the same RG procedure as (27), we obtain

$$\delta\rho_e = \frac{3\pi^2}{4k^2}\frac{n_i}{n_e}\rho_e^{(0)}\left[1 - k\zeta\left(\frac{T}{T_K}\right)^{\Delta}\right] \tag{31}$$

where $\rho_e^{(0)} = 4\pi/e^2k_F$ is the resistivity in the unitary limit and k_F denotes the Fermi wavevector which is related to the conduction electron density of states through $\rho = k_F m_e/2\pi^2$. The $T = 0$ value of the resistivity and the exponent Δ have been reported [11] and an exact expression for the resistivity for $k = 2$ up to a constant factor ζ has been derived recently [16]. As in the exactly screened Kondo problem, the corresponding resistivity decreases upon increasing temperature.

An important point is that, even at $T = 0$, there is still inelastic scattering. In other words, the impurity spin together with its asymptotical screening cloud (see the magnetization section) cannot be regarded as inert. Let us see how the assumption of elastic scattering only at $T = 0$ would run into trouble. Following Nozières [19], the total resistivity can be divided into elastic and inelastic parts:

$$\delta\rho_e = \delta\rho_e^{el} + \cos(2\phi_0)\delta\rho_e^{in} \tag{32}$$

where ϕ_0 is the scattering phase shift at the Fermi energy. This separation is valid at least for weak inelastic scattering, which would indeed be the case at low T if only elastic scattering were present at $T = 0$. For elastic scattering, we could identify

$$\rho\Sigma''(\omega=0, T=0) \sim n_i \frac{S(S+1)}{\pi} \sin^2\phi_0$$

which would lead to

$$\phi_0 \sim \frac{\pi}{k}. \tag{33}$$

The resistivity due to the elastic scattering can be calculated by using (27) as the scattering rate. To leading order

$$\delta\rho_e^{el}(T) = \int_0^\infty \frac{d\omega}{2T\cosh^2\omega/2T} \frac{3\pi^2 n_i}{4k^2 n_e} \rho_e^{(0)} \left[1 - A_0\left(\frac{\omega}{T_K}\right)^\Delta\right]. \tag{34}$$

Substituting (31), (33) and (34) into (32), we would obtain the inelastic contribution to the resistivity, $\delta\rho_e^{in} < 0$, which certainly is impossible. This contradiction indicates that there is both elastic and inelastic scattering at $T = 0$. For the two-channel case, $k = 2$, it has been suggested that there is actually only inelastic scattering [16].

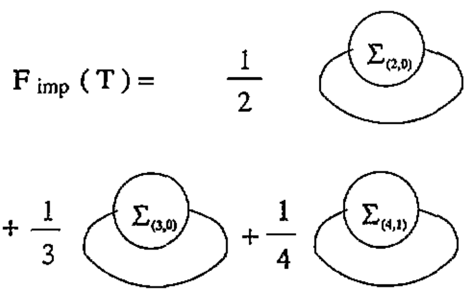


Figure 2. Diagrams for the free energy. The full curve represents the bare conduction electron propagator. The discs encircling the self-energy symbols stand for the conduction electron self-energies.

6. Specific heat and entropy

To calculate the free energy, we can use the linked cluster theorem:

$$F = F_0 - \sum_{n=2}^{\infty} U_n \tag{35}$$

where F_0 is the free energy of the non-interacting Fermi sea and a decoupled free impurity spin. The diagrams for the first three U_n are drawn in figure 2. The results, after completing the Matsubara frequency summations, are

$$U_2 = -\frac{3n_{\rm i}}{16} g^2 k \int {\rm d}\epsilon\, {\rm d}\epsilon_1 \frac{f(\epsilon) - f(\epsilon_1)}{\epsilon - \epsilon_1} \tag{36}$$

$$U_3 = \frac{n_{\rm i}}{8} g^3 k \int {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3 \left(\frac{f(\epsilon_1) f(\epsilon_2) f(-\epsilon_3)}{(\epsilon_3 - \epsilon_1)(\epsilon_3 - \epsilon_2)} + 2\,\text{permutations} \right) \tag{37}$$

$$U_4 = \frac{3n_{\rm i}}{64} g^4 k^2 \int {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3\, {\rm d}\epsilon_4\, f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4) \times \left[\frac{1}{(\epsilon_1 - \epsilon_2)(\epsilon_3 - \epsilon_4)} \left(\frac{1 - {\rm e}^{\beta(\epsilon_2 - \epsilon_1 + \epsilon_3 - \epsilon_4)}}{\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4} + \frac{{\rm e}^{\beta(\epsilon_2 - \epsilon_1)} - {\rm e}^{\beta(\epsilon_3 - \epsilon_4)}}{\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4} \right) \right]. \tag{38}$$

As noted by Kondo [35], it is quite delicate to extract the linear-T terms in (37) and (38). Here we follow the method used by Kondo and present the calculations only for the integrals not calculated in [35].

The calculation of U_2 is simple and straightforward. The result is a constant plus T^2 corrections. To calculate U_3, we define a new integral I_3:

$$I_3 = \int_{-D}^{D} {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3 \frac{f(\epsilon_1) f(\epsilon_2) f(-\epsilon_3)}{(\epsilon_3 - \epsilon_1)_\delta (\epsilon_3 - \epsilon_2)_\delta}$$

where we have introduced the notation

$$\frac{1}{(x)_\delta} = \frac{x}{x^2 + \delta^2} \qquad \delta \to 0.$$

The $\delta \to 0$ limit is taken after the integrations are completed. Kondo [35] showed

$$U_3 = \frac{3n_{\rm i}}{8} g^3 k (I_3 - \tfrac{1}{6}\pi^2 T) \tag{39}$$

so we only sketch the calculation for I_3:

$$I_3 = \int {\rm d}\epsilon_1\, {\rm d}\epsilon_2 f(\epsilon_1) f(\epsilon_2) \int_{-D}^{D} \frac{{\rm d}\epsilon_3}{(\epsilon_3 - \epsilon_1)_\delta (\epsilon_3 - \epsilon_2)_\delta} - \int {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3 \frac{f(\epsilon_1) f(\epsilon_2) f(\epsilon_3)}{(\epsilon_3 - \epsilon_1)_\delta (\epsilon_3 - \epsilon_2)_\delta}.$$

The first integral has no linear-T contribution which can be directly checked. Thus, up to a constant term, we have

$$\begin{aligned} I_3 &= -\frac{1}{3} \int_{-D}^{D} {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3\, f(\epsilon_1) f(\epsilon_2) f(\epsilon_3) \\ &\quad \times \left(\frac{1}{(\epsilon_3 - \epsilon_1)_\delta (\epsilon_3 - \epsilon_2)_\delta} + \frac{1}{(\epsilon_1 - \epsilon_2)_\delta (\epsilon_1 - \epsilon_3)_\delta} + \frac{1}{(\epsilon_2 - \epsilon_1)_\delta (\epsilon_2 - \epsilon_3)_\delta} \right) \\ &= -\frac{1}{3} \int_{-D}^{D} {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3 \frac{f(\epsilon_1)\, f(\epsilon_2)\, f(\epsilon_3) [(\epsilon_1 - \epsilon_2)^2 + (\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3)]\delta^2}{[(\epsilon_1 - \epsilon_2)^2 + \delta^2][(\epsilon_1 - \epsilon_3)^2 + \delta^2][(\epsilon_2 - \epsilon_3)^2 + \delta^2]} \\ &= -\frac{\pi^2}{3} \int_{-D}^{D} {\rm d}\epsilon_1\, {\rm d}\epsilon_2\, {\rm d}\epsilon_3 f(\epsilon_1) f(\epsilon_2) f(\epsilon_3) \delta(\epsilon_1 - \epsilon_3) \delta(\epsilon_2 - \epsilon_3) \\ &= -\frac{\pi^2}{3} \int_{-D}^{D} {\rm d}\epsilon_1\, [f(\epsilon_1)]^3 = -\tfrac{1}{2}\pi^2 T. \end{aligned}$$

Combining the above result with (39), we obtain

$$U_3 = \text{constant} - \tfrac{1}{4} n_i \pi^2 g^3 k T. \tag{40}$$

Next we turn to U_4, which is calculated in a similar way. Defining a new function

$$\begin{aligned} I_4 = \int & \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \mathrm{d}\epsilon_3\, \mathrm{d}\epsilon_4\, f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4) \\ & \times \left[\frac{1}{(\epsilon_1 - \epsilon_2)_\delta (\epsilon_3 - \epsilon_4)_\delta} \left(\frac{1 - \mathrm{e}^{\beta(\epsilon_2 - \epsilon_1 + \epsilon_3 - \epsilon_4)}}{(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)_\delta} + \frac{\mathrm{e}^{\beta(\epsilon_2 - \epsilon_1)} - \mathrm{e}^{\beta(\epsilon_3 - \epsilon_4)}}{(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)_\delta} \right) \right] \\ = 4 \int & \mathrm{d}\epsilon_1\, \mathrm{d}\epsilon_2\, \mathrm{d}\epsilon_3\, \mathrm{d}\epsilon_4\, \frac{f(-\epsilon_1) f(\epsilon_2) f(\epsilon_3) f(-\epsilon_4)}{(\epsilon_1 - \epsilon_2)_\delta (\epsilon_3 - \epsilon_4)_\delta (\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)_\delta} \end{aligned}$$

then, following the same steps Kondo employed to prove (39), it is straightforward to show that

$$U_4 - \frac{3n_i}{64} g^4 k^2 I_4 = \frac{3\pi^2 n_i}{32} g^4 k^2 T. \tag{41}$$

It can be further verified that I_4 has no contribution linear in T. Therefore

$$U_4 = \text{constant} + \frac{3\pi^2 n_i}{32} g^4 k^2 T. \tag{42}$$

Substituting (40) and (42) into (35) and setting $n_i = 1$ for simplicity, we obtain the free energy shift due to the presence of the impurity spin:

$$F_{\mathrm{imp}}(T) = -E_0 - T \ln 2 + \tfrac{1}{4}\pi^2 T (k g^3 - \tfrac{3}{8} k^2 g^4) + \mathcal{O}(g^4 k) \tag{43}$$

where E_0 is the ground-state energy.

Since the free energy shift is RG invariant up to an additive constant, we obtain the scaling solution for the free energy shift at $T \to 0$ after carrying out the standard RG procedure

$$\begin{aligned} F_{\mathrm{imp}}(T) &= -E_0' - T \ln 2 + \tfrac{1}{4}\pi^2 T \big(k g_R^3(T) - \tfrac{3}{8} k^2 g_R^4(T) \big) \\ &= -E_0' - T \left(\ln 2 - \frac{\pi^2}{2k^2} \right) - \tfrac{3}{4}\pi^2 \zeta^2 T \left(\frac{T}{T_K} \right)^{2\Delta} + \mathcal{O}[T (T/T_K)^{3\Delta}] \end{aligned} \tag{44}$$

where E_0' is in general different from the E_0 of (43). The impurity specific heat is

$$C_{\mathrm{imp}} = -T \frac{\partial^2 F_{\mathrm{imp}}}{\partial T^2} \simeq \tfrac{3}{2}\pi^2 \zeta^2 \Delta \left(\frac{T}{T_K} \right)^{2\Delta}. \tag{45}$$

The critical exponent, $\alpha = 2\Delta$, agrees with the previous result [9, 25]. The impurity entropy is reduced to a universal value [10]

$$S_{\mathrm{imp}}(T = 0) = \ln 2 - \frac{\pi^2}{2k^2} \simeq \ln\left(2 - \frac{\pi^2}{k^2} \right). \tag{46}$$

Actually, the correction to the entropy occurs only when one first takes the thermodynamic limit $\mathcal{N} \to \infty$ and then the limit $T \to 0$. For a finite system, the integrals in U_3 and U_4 should be replaced by discrete momentum summations over the Brillouin zone, which will not give rise to contributions linear in T to the free energy, and thus no correction to the bare $\ln 2$ term of the entropy. Therefore, the finite system always remains doubly degenerate.

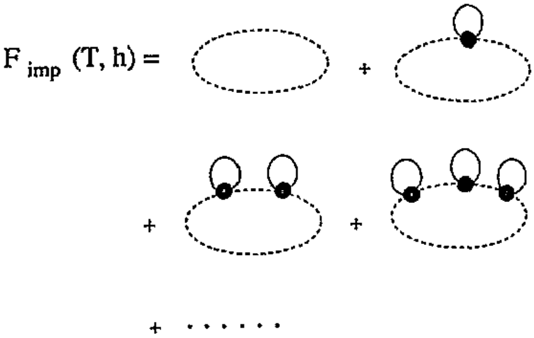


Figure 3. Diagrams for the free energy in the magnetic field.

7. Field-dependent magnetization

A natural question following the finite entropy is whether or not there is a corresponding residual impurity spin. To answer this question we calculate the field-dependent impurity magnetization defined as the total magnetization of the system subtracting the free Fermi sea contribution. In the presence of an external magnetic field, the leading-order diagrams to the free energy are shown in figure 3. This is an infinite series. The diagram with l vertices has a contribution

$$\delta F_l = -\frac{1}{\beta}\frac{k^l}{l}(-1)^{l+1}\left(\frac{-J}{\mathcal{N}\beta}\sum_{k,\omega_n}\sum_{\sigma=\pm}\frac{-\sigma/2}{\mathrm{i}\omega_n-\epsilon_k-\sigma h}\right)^l\sum_{\omega_m,\sigma'=\pm}\left(\frac{-\sigma'/2}{\mathrm{i}\omega_m-\mathrm{i}\omega_0-\sigma'\mu_S h}\right)^l. \tag{47}$$

The momentum summation over the conduction band is readily carried out:

$$\frac{J}{\mathcal{N}\beta}\sum_{k,\omega_n}\sum_{\sigma=\pm}\frac{\sigma}{\mathrm{i}\omega_n-\epsilon_k-\sigma h}=g\int_{-D}^{D}\mathrm{d}\epsilon[f(\epsilon+h)-f(\epsilon-h)]=-2gh \tag{48}$$

where again $f(\epsilon)$ is the Fermi–Dirac function. Summing up all the diagrams in figure 3 and using (48), the free energy shift due to the impurity spin is

$$F_{\mathrm{imp}}(T,h)=\sum_l\delta F_l=\sum_{\sigma=\pm}\sum_{l=1}^{\infty}\left(\frac{-\sigma gkh}{2}\right)^l\frac{1}{l!}\left[\frac{d^{l-1}f(z)}{\mathrm{d}z^{l-1}}\right]_{z=\mathrm{i}\omega_0+\sigma\mu_S h}.$$

Introducing the primitive of $f(z)$: $\mathrm{d}u(z)/\mathrm{d}z=f(z)$, the series can be summed:

$$F_{\mathrm{imp}}(T,h)=\sum_{\sigma=\pm}\left[u(\mathrm{i}\omega_0+\sigma\mu_S h-\sigma kgh/2)-u(\mathrm{i}\omega_0+\sigma\mu_S h)\right].$$

The corresponding magnetization is

$$\delta M(T,h)=-\frac{\partial F_{\mathrm{imp}}}{\partial h}=\left(\mu_S-\frac{kg}{2}\right)\tanh\left[\beta h\left(\mu_S-\frac{kg}{2}\right)\right]-\tanh(\beta\mu_S h).$$

Adding to the above result the bare magnetization of the impurity spin, which just cancels the second term of the above expression, we finally obtain

$$M(T,h)=\left(\mu_S-\frac{kg}{2}\right)\tanh\left[\beta h\left(\mu_S-\frac{kg}{2}\right)\right]. \tag{49}$$

This is the leading-order temperature- and field-dependent magnetization.

In this section, we shall mainly consider the case $\mu_S = 1$, i.e. an equal gyromagnetic ratio. Expression (49) has the form characteristic of a reduced free spin:

$$M = S_{\rm eff} \tanh(S_{\rm eff} h/T) \qquad S_{\rm eff} = 1 - kg/2 = (g^* - g)k/2.$$

The fact that $S_{\rm eff} \to 0$ following a power law, as seen from (25), implies that the impurity spin is completely screened only for the infinite system. A finite system of size L behaves as if there is a partially screened spin $S_{\rm eff} \sim L^{-\Delta}$ since there is a minimum energy unit $1/L$ to cut off the scaling. The asymptotic screening also suggests that the ground-state degeneracy will be lifted by any small magnetic field at $T = 0$. Using the Maxwell relation $\partial S/\partial h = \partial M/\partial T$, the leading-order field-dependent entropy is

$$\begin{aligned} S_{\rm imp}(T, h) &= \ln 2 + \int_0^h {\rm d}h' \frac{\partial M(T, h')}{\partial T} \\ &= \ln 2 - \int_0^{\beta h|1-gk/2|} {\rm d}x \frac{x}{\cosh^2 x} \to 0 \qquad \text{for} \quad T \to 0 \quad h \neq 0. \end{aligned} \tag{50}$$

We expect that this is true to any order. In particular, the entropy change, $\pi^2/(2k^2)$, in the next order (46) will be removed by the magnetic field.

From (49), we obtain the low-temperature magnetic susceptibility and its scaling form [9, 25]:

$$\chi_{\rm imp}(T) = \beta\left(1 - \frac{kg}{2}\right)^2 = \beta\left(1 - \frac{kg_{\rm R}(T)}{2}\right)^2 \overset{T<T_{\rm K}}{=} \left(\frac{k\zeta}{2}\right)^2 \frac{1}{T}\left(\frac{T}{T_{\rm K}}\right)^{2\Delta} \tag{51}$$

indicating that the fixed point is a spin singlet since $S_{\rm eff}^2 \sim T\chi_{\rm imp}(T) \sim T^{2\Delta} \to 0$ as $T \to 0$. The spin is quenched very slowly compared with $S_{\rm eff}^2 \sim T$ in the exactly screened case. Also from (49), we determine the field dependence of the zero-temperature magnetization [9]:

$$M(T = 0, h) = 1 - \tfrac{1}{2}kg = 1 - \frac{kg_{\rm R}(h)}{2} \overset{T<T_{\rm K}}{=} \tfrac{1}{2}k\zeta(h/T_{\rm K})^{\Delta}. \tag{52}$$

This shows that the spin operator has the scaling dimension Δ.

Using the well known results for the bulk specific heat and magnetic susceptibility, $C_{\rm bulk} = 2k\pi^2 T\rho/3$ and $\chi_{\rm bulk} = 2k\rho$ respectively [36], the leading-order Wilson ratio is determined from (45) and (51):

$$W = \frac{\chi_{\rm imp}}{C_{\rm imp}} \frac{C_{\rm bulk}}{\chi_{\rm bulk}} = \frac{(\frac{1}{2}k\zeta)^2(1/T)(T/T_{\rm K})^{2\Delta}}{\frac{3}{2}\pi^2\zeta^2\Delta(T/T_{\rm K})^{2\Delta}} \frac{\frac{2}{3}k\pi^2 T\rho}{2k\rho} = \frac{k^3}{36} \tag{53}$$

in agreement with the conformal field result [9]. The Wilson ratio is universal because the only parameter having a possible dependence on the cut-off scheme is ζ, which cancels out.

For $\mu_S \neq 1$, the external magnetic field couples to an unconserved spin operator. The magnetization will acquire an anomalous dimension and we shall discuss it in detail in section 9.

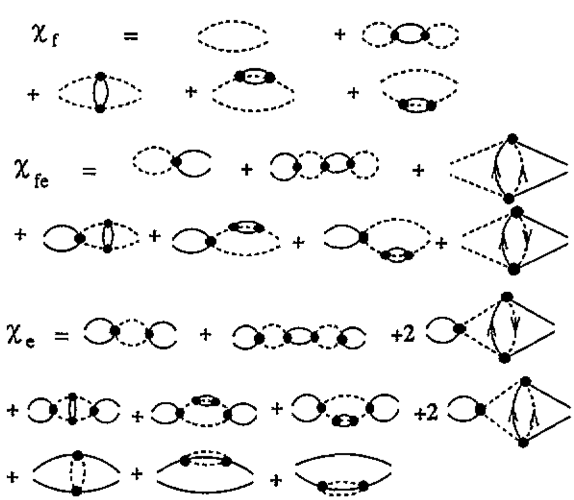


Figure 4. Diagrams for three dynamical spin correlation functions.

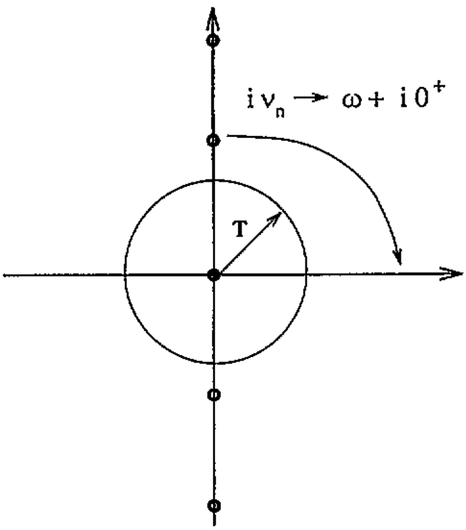


Figure 5. Analytic continuation in the complex frequency plane carried out in the region outside a disc of radius of order T.

8. Dynamical susceptibility

For the one-channel Kondo problem, it is well known that the impurity spin flips at a typical rate of the Kondo temperature T_K. At $T < T_K$, the impurity spin is effectively inert leading to a fixed point of local Fermi-liquid type described by a phase shift. To understand the multichannel case, we calculate the dynamical spin susceptibility. As we shall see, the typical spin flipping rate is given by the temperature.

We proceed as usual by first calculating the Matsubara spin–spin correlation function:

$$\chi_f(\tau) = \langle \hat{\mathrm{T}} \boldsymbol{S}(\tau) \cdot \boldsymbol{S}(0) \rangle = \frac{1}{\beta} \sum_n \chi_f(\mathrm{i}\nu_n)\, \mathrm{e}^{-\mathrm{i}\nu_n \tau} \tag{54}$$

where $\nu_n = 2n\pi/\beta$. To sub-leading order, the diagrams are shown in figure 4. The result of the calculation of these diagrams is

$$\chi_f(\mathrm{i}\nu_n) = \delta_{n,0}\tfrac{3}{4}\beta\left[1 - g^2k(\ln\beta D - \ln 2 + I_0)\right] - \tfrac{3}{4}g^2kK(\mathrm{i}\nu_n) \tag{55}$$

where $I_0 = \int_0^\infty \mathrm{d}x[\tanh(x/2)/x - 1/(1+x)] \simeq 0.125$ and

$$K(\mathrm{i}\nu_n) = \int_{-D}^{D}\int_{-D}^{D} \frac{\mathrm{d}\epsilon_1\,\mathrm{d}\epsilon_2}{(\epsilon_1-\epsilon_2)^2+\nu_n^2}\left(\frac{f(\epsilon_1)-f(\epsilon_2)}{\epsilon_1-\epsilon_2} - f'(\epsilon_1)\delta_{n,0}\right)$$
$$= -\frac{\pi}{|\nu_n|}(1-\delta_{n,0}) + \mathcal{O}(D^{-1}). \tag{56}$$

To understand the result just derived, let us consider $\chi_f(z)$ in the complex frequency plane $\mathrm{i}\nu_n \to z$ outside a disc of radius of order T (figure 5). Knowing its values at a set of discrete points on the imaginary axis, $z = \mathrm{i}\nu_n$, is enough to give a unique analytic continuation:

$$\chi_f(\mathrm{i}\nu_n) = \tfrac{3}{4}\pi g^2 k\frac{1}{|\nu_n|} \to \tfrac{3}{4}\pi g^2 k\frac{\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\,z)}{z} \tag{57}$$

where $\mathrm{sgn}(\mathrm{Im}\,z)$ means taking the sign of the imaginary part of z. When z approaches the real axis, $z = \omega + \mathrm{i}\delta$, $\chi_f(\omega+\mathrm{i}\delta) = \chi_f'(\omega) + \mathrm{i}\,\mathrm{sgn}(\delta)\chi_f''(\omega)$, and we find

$$\chi_f''(\omega) = \tfrac{3}{4}\pi g^2 k\frac{1}{\omega}. \tag{58}$$

This is valid in the whole plane outside a disc of radius of order T. There is no energy scale to cut off $1/\omega$ dependence. Certainly for $\omega \ll T$, this dependence should be flattened out as seen from the $\nu_n = 0$ part of (55). If we recall that in the exactly screened Kondo problem ($k = 2S$), $\chi_f''(\omega)/\omega \sim 1/(\omega^2 + T_K^2)$, we see that $\chi_f''(\omega) \sim 1/\omega$, similar to (58) only when $\omega > T_K$. Since $\chi_f''(\omega)/\omega$ is the spectral distribution function of the susceptibility, the system gives sizable magnetic response only when all the contributions up to the energy scale T_K are included. In other words, the impurity spin is effectively quenched on a timescale longer than $1/T_K$. In the multichannel case, the role of T_K is played by the temperature T! The impurity is only marginally quenched since the timescale is $1/T$.

In the other limit, $\omega \ll T$, carrying out the analytic continuation as in [15], we find

$$\frac{\chi_f''(\omega, T)}{\omega} \sim T^{2\Delta-2}. \tag{59}$$

Note that if $\mu_S \gg 1$, the coupling of the nuclear magnetic moment to the impurity spin dominates and (59) is proportional to the NMR relaxation rate $1/T_1T$. In the limit $k \gg 1$, a local probe basically sees a nearly free impurity spin.

In the two-channel case, numerical work in the non-crossing approximation [7] and the solution at a special Toulouse point [14] have found $\chi_f''(\omega) \sim \tanh(\omega/2T)/(\omega^2 + T_K^2)$. The $\tanh(\omega/2T)$ term has also been produced by the conformal field theory method, and is a property of the fixed point itself. It would be interesting to see if the remaining Lorentzian form with width T_K can be reproduced from the perturbation of the leading irrelevant operator in the exact conformal field theory calculations. For $k > 2$, we do not expect a Lorentzian form with finite width T_K for the following reason. The multiplicative renormalization factor (Z_h^2 of (68) at $\mu_S = \infty$, see the next section) for χ_f will bring in an anomalous dimension $\omega^{2\Delta}$ to (58). We see that $\chi_f''(\omega) \sim \omega^{2\Delta-1}$ for $T < \omega < T_K$. Since $2\Delta = 4/(k+2) < 1$, this frequency dependence cannot be reconciled with a Lorentzian form with a finite width.

9. Magnetic susceptibility for $\mu_S \neq 1$

As well as the time-ordered impurity spin–spin correlation function studied in the last section, we can also define

$$\chi_{fe}(q,\tau) = \langle \hat{T} S_e(q,\tau) \cdot S(\tau=0) \rangle = \frac{1}{\beta} \sum_n \chi_{fe}(q, i\nu_n) e^{-i\nu_n \tau} \tag{60}$$

$$\chi_e(q,\tau) = \langle \hat{T} S_e(q,\tau) \cdot S_e(0,0) \rangle = \frac{1}{\beta} \sum_n \chi_e(q, i\nu_n) e^{-i\nu_n \tau} \tag{61}$$

where the conduction electron spin operator is defined as

$$S_e(q) = \frac{1}{2} \sum_k \sum_{\substack{\mu,\nu=\pm \\ \lambda=1,\dots,k}} c^\dagger_{k,\mu\lambda} \sigma_{\mu\nu} c_{k+q,\nu\lambda}. \tag{62}$$

The Feynman diagrams for these two correlation functions are shown in figure 4 and calculated in appendix B.

When the impurity spin has a different gyromagnetic ratio from the conduction electrons, a uniform external magnetic field couples to an unconserved spin operator $S_h = S_e(q = 0) + \mu_S S$. The magnetic susceptibility is then only RG invariant up to a multiplicative renormalization factor

$$\chi_{imp}(T, g, D) = \left[Z_h\big(g, g_R(D')\big)\right]^2 \chi_{imp}\big(T, g_R(D'), D'\big) \tag{63}$$

where Z_h is the multiplicative renormalization factor for the operator S_h and we recall $g_R(D) = g$. If $\mu_S = 1$, then $Z_h \equiv 1$. The RG equation for χ is

$$\left[D\frac{\partial}{\partial D} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_h(g)\right]\chi_{imp}(T, g, D) = 0 \tag{64}$$

$$\gamma_h(g) = D\frac{\partial \ln Z_h}{\partial D}. \tag{65}$$

The perturbative result for χ to the sub-leading order can be found from the results of three dynamical correlation functions calculated in the last section and appendix B:

$$\begin{aligned}\chi_{imp}(T, g, D) &= \tfrac{4}{3}\left[\chi_e(q=0, \nu_n=0) + 2\mu_S \chi_{fe}(q=0, \nu_n=0) + \mu_S^2 \chi_f(\nu_n=0)\right] \\ &= \beta\{(\mu_S - \tfrac{1}{2}gk)^2[1 - g^2k(\ln\beta D - \ln 2 + I_0)] - g^2k(\mu_S - \tfrac{1}{2}gk)\ln 2\}.\end{aligned} \tag{66}$$

The leading term of $\gamma_h(g)$ can be obtained from (66) using the same method of section 4:

$$\gamma_h(g) = \frac{\mu_S - 1}{2\mu_S - gk} g^2 k. \tag{67}$$

Its value at the fixed point, $\gamma_h(g^*)$, gives the anomalous dimension of the operator S_h. The anomalous dimensions for the operators S and $S_e(q = 0)$ are the limiting values at $\mu_S \to \infty$ and $\mu_S \to 0$ respectively.

To obtain the scaling solution, we first find the multiplicative renormalization factor

$$Z_h(g, g_R(D')) = \exp\left(\int_g^{g_R(D')} dg' \frac{\gamma_h(g')}{\beta(g')}\right) = \left(\frac{gk - 2\mu_S}{gk - 2}\right)^2 \left(\frac{g_R(D')k - 2}{g_R(D')k - 2\mu_S}\right)^2. \tag{68}$$

Setting $D' = T$ in (63) and (68), we obtain the leading-order susceptibility (since we only have the leading-order Z_h):

$$\chi_{\rm imp}(T, g, D) = \left(\frac{gk - 2\mu_S}{gk - 2}\right)^2 [\chi_{\rm imp}(T, g, D)]_{\mu_S = 1}. \tag{69}$$

This is the main result of this section: the static magnetic response is the same, up to a constant factor, whether or not the conduction electrons and the impurity spin have the same gyromagnetic ratio. I believe that this conclusion remains true for all overscreened Kondo problems, and even for the exactly screened case of $k = 1$. If the initial starting point belongs to the weak coupling $g \to 0$

$$\chi_{\rm imp}(T, g, D) = \mu_S^2 [\chi_{\rm imp}(T, g, D)]_{\mu_S = 1}. \tag{70}$$

This is the well known result for all kinds of Kondo problem: the contribution to the magnetization from the conduction electrons is suppressed by a factor of T_K/D compared with that of the impurity spin [37]. Extending the treatment in this section to the field-dependent magnetization and dynamical susceptibilities is straightforward.

10. Knight shift

The Knight shift gives a direct measurement of the spatial structure of the conduction electron screening cloud. Considering an impurity spin sitting at the origin and a uniform magnetic field h being applied to the system, the Knight shift measures the magnetization in the space surrounding the impurity spin:

$$M(\boldsymbol{r}) = \sum_{\substack{\sigma=\pm \\ \lambda=1\ldots k}} \sigma \langle c^\dagger_{\sigma,\lambda}(\boldsymbol{r}) c_{\sigma,\lambda}(\boldsymbol{r})\rangle = \chi(\boldsymbol{r}) h \qquad \chi(\boldsymbol{r}) = \frac{1}{\mathcal{N}} \sum_{\boldsymbol{q}} \chi(\boldsymbol{q}) {\rm e}^{{\rm i}\boldsymbol{q}\cdot\boldsymbol{r}}. \tag{71}$$

We implicitly assume that the free Fermi sea contribution to $M(\boldsymbol{r})$ has been subtracted. The result for the Knight shift is contained in the two dynamical correlation functions defined in the last section:

$$\chi(\boldsymbol{q}) = \tfrac{4}{3} [\chi_e(\boldsymbol{q}, {\rm i}\nu_n = 0) + \mu_S \chi_{fe}(\boldsymbol{q}, {\rm i}\nu_n = 0)]. \tag{72}$$

Using the results of appendix B, we find to sub-leading order in $1/k$

$$\begin{aligned}\chi(\boldsymbol{q}) = &-\frac{gk}{2T}\Pi_0(\boldsymbol{q})\{(\mu_S - \tfrac{1}{2}gk)[1 - g^2k(\ln D/T - \ln 2 + I_0)] - \tfrac{1}{2}\ln 2 g^2 k\} \\ &- \frac{\ln 2}{2T} g^2 k(\mu_S - \tfrac{1}{2}gk)\Pi_1(\boldsymbol{q})\end{aligned} \tag{73}$$

$$\Pi_0(\boldsymbol{q}) = -\frac{1}{\rho\mathcal{N}} \sum_{\boldsymbol{k}} \frac{f(\epsilon_{\boldsymbol{k}}) - f(\epsilon_{\boldsymbol{k}+\boldsymbol{q}})}{\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}+\boldsymbol{q}}} \tag{74}$$

$$\Pi_1(\boldsymbol{q}) = \frac{1}{2\ln 2}\frac{1}{\rho^2\mathcal{N}^2} \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{\tanh(\epsilon_{\boldsymbol{k}}/2T)}{\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}'}} \left(\frac{f(\epsilon_{\boldsymbol{k}}) - f(\epsilon_{\boldsymbol{k}'+\boldsymbol{q}})}{\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}'+\boldsymbol{q}}} - \frac{f(\epsilon_{\boldsymbol{k}'}) - f(\epsilon_{\boldsymbol{k}'+\boldsymbol{q}})}{\epsilon_{\boldsymbol{k}'} - \epsilon_{\boldsymbol{k}'+\boldsymbol{q}}}\right) \tag{75}$$

where $I_0 \simeq 0.125$, is given in the last section. We have included proper normalization factors in the definitions of $\Pi_0(\boldsymbol{q})$ and $\Pi_1(\boldsymbol{q})$ such that $\Pi_0(\boldsymbol{q} = 0) = 1$ and $\Pi_1(\boldsymbol{q} = 0) = 1$.

Note that the Fourier transform of $\Pi_0(q)$ has the well known Friedel oscillation form in space [38]. The Fourier transform of $\Pi_1(q)$ should give similar spatial variation. From the rotational symmetry, $\chi(\boldsymbol{q})$ can only depend on $q = |\boldsymbol{q}|$. Furthermore, it is not difficult to see that the two dimensionless functions Π_0 and Π_1 are only functions of q/k_{F}. Their Fourier transforms only depend on $k_{\mathrm{F}}r$, where r is the distance to the impurity spin.

The important feature of (73) is its logarithmic singularity in the temperature T, no matter what value is taken by q. One can also explicitly verify that $\chi(q)$ satisfies the RG equation:

$$\left[D\frac{\partial}{\partial D} + \beta(g)\frac{\partial}{\partial g} + \gamma_h(g) + [\gamma_h(g)]_{\mu_S=0}\right]\chi\left(\frac{q}{k_{\mathrm{F}}}, \frac{T}{D}, g\right) = 0 \tag{76}$$

where $\beta(g)$ and $\gamma_h(g)$ are given by (20) and (67) respectively. As can be seen from (73), the actual functional form of $\chi(q)$ is $\chi(q, T, D, g) = \widetilde{\chi}(q/k_{\mathrm{F}}, T/D, g)/T$ where $\widetilde{\chi}$ is a dimensionless function. Nevertheless, the simplified notation of the functional dependence of $\chi(q)$ in (76) is harmless. The scaling solution can be found in a similar way to the last section:

$$\chi\left(\frac{q}{k_{\mathrm{F}}}, \frac{T}{D}, g\right) = Z_h(g, g_{\mathrm{R}}(T))\big[Z_h(g, g_{\mathrm{R}}(T))\big]_{\mu_S=0}\chi\left(\frac{q}{k_{\mathrm{F}}}, 1, g_{\mathrm{R}}(T)\right). \tag{77}$$

Using the results for Z_h and g_{R}, we obtain to leading order

$$\chi\left(\frac{q}{k_{\mathrm{F}}}, \frac{T}{D}, g\right) = \frac{gk(gk - 2\mu_S)}{(gk-2)^2}\chi_{\mathrm{imp}}(T)\,\Pi_0(q) \tag{78}$$

where $\chi_{\mathrm{imp}}(T)$ is the uniform susceptibility found in section 7. A knowledge of Z_h to sub-leading order could extend (78) to the next order. When Fourier transforming (78) to real space, $M(r)$ is a function of $k_{\mathrm{F}}r$. So the only length scale is $1/k_{\mathrm{F}}$. Naively, the appearance of a new energy scale T_{K} would imply a new length scale $v_{\mathrm{F}}/T_{\mathrm{K}}$. Thinking over why it did not appear, we notice that a new length scale $v_{\mathrm{F}}/T_{\mathrm{K}} \gg 1/k_{\mathrm{F}}$ would appear only if the logarithmic singularity were cut off by $v_{\mathrm{F}}q$ when $v_{\mathrm{F}}q > T$. This does not happen! For $k_{\mathrm{F}} > q \gg T_{\mathrm{K}}/v_{\mathrm{F}}$, we essentially look at the conduction electron spin polarization only a few lattice spacings away from the impurity spin. If there were a large screening cloud with a length scale $v_{\mathrm{F}}/T_{\mathrm{K}}$, the screening of the impurity spin would not be complete inside the screening cloud. This would imply that there is no Kondo effect for $q \gg T_{\mathrm{K}}/v_{\mathrm{F}}$ even if $T/T_{\mathrm{K}} \to 0$. On the contrary, we always find the $\ln T$ singularity which implies Kondo screening. The length scale of the screening cloud is $1/k_{\mathrm{F}}$. This tells us that the multichannel system is critical only in the time direction, not in the space direction.

Most generally, $\chi(r)$ of (71) contains a non-oscillating part and an $\mathrm{e}^{\mathrm{i}2k_{\mathrm{F}}r}$ oscillating part for which the q dependences of $\chi(q)$ near $q \sim 0$ and $q \sim 2k_{\mathrm{F}}$ are responsible. The spatial variation of the non-oscillating part is simple because $\Pi_0(q)$, $\Pi_1(q)$ and therefore $\chi(q)$ all have a well behaved q dependence near $q \sim 0$, and we can extend the above conclusion of only one length scale $1/k_{\mathrm{F}}$ to all kinds of Kondo problem. In the $k = 1$ case, the effective Kondo interaction flows to strong coupling at low energy. From the renormalization group point of view, it means that summing up leading or sub-leading logarithmic series is no longer enough to reach an energy scale below the Kondo temperature. But organizing the perturbative expansion in the coupling constant g into successive logarithmic series (the first term of each series has successively higher powers in g) according to the renormalization group is still possible. This is equivalent to writing $\beta(g)$ as an expansion in g. The occurrence of a strong-coupling fixed point means that we need all terms in the expansion of $\beta(g)$ to describe the low-energy behaviour. Supposing we could find all of them, we would have a correct low-energy theory by summing up all the infinite number of terms. The fact is that the infrared singularity is not cut off by $v_F q$. In other words, the logarithmic series are in $\ln T$ but not in $\ln q$. This fact does *not* change merely because we have to include high-order logarithmic series. The mathematical expression of this fact is the scaling equation (76) whose validity is not expected to be affected. The difference between all kinds of Kondo problem, either overscreened, exactly screened or underscreened, amounts to the different behaviour of $g_R(T)$ and Z_h. In the scaling equation (76), the parameter q/k_F is just an idle spectator of the screening process. The right-hand side of (77) only depends on q/k_F, T/T_K and possibly the initial coupling constant due to Z_h. We see that v_F/T_K will never have the chance to appear as a length scale. Certainly, if the renormalization of an effective interaction to strong coupling leads to a phase transition, the validity of the scaling equation (76) may be affected by passing the transition. However, the impurity problem has a dimension $0+1$, prohibiting a phase transition (a first-order transition is possible, but we know it is not there). Thus, our calculation provides an explicit analytical demonstration that the screening cloud has a spatial size of order $1/k_F$, and agrees with the well known experimental result [28].

11. Discussion

We have carried out a comprehensive study on the non-Fermi-liquid fixed point of the multichannel Kondo model emphasizing several intriguing aspects. As the actual realization of this model in heavy-fermion and metallic glassy systems is still an open question, a thorough understanding of the model should be very helpful in devising new experimental tests and making a comparison with experimental results. Part of the driving force behind the recent resurgence of interest is the resemblance of its low-energy behaviour to the normal-state properties of high-T_c cuprates. This similarity has been further exploited recently [6] arguing that a situation similar to the two-channel Kondo model is realized in the copper-oxide plane.

An interesting follow-up subject is to study the possibility of the evolution of the local non-Fermi-liquid fixed point into a coherent lattice one in an overscreened Kondo lattice, especially in two or three dimensions. The first step is to study the two-channel–two-impurity problem. This has been carried out using the numerical renormalization group method [39]. The one-impurity fixed point is found to be unstable against developing correlations between the impurity spins. This is expected in view of the asymptotical screening of the impurity spin and the residual entropy. One artificial way to suppress magnetic correlations between impurity spins is to go to infinite dimensions. Unfortunately, the presence of finite degeneracy on each site prevents development of true coherence. In any realistic situation, non-trivial magnetic correlations must intervene to lift the residual degeneracy of the impurity fixed point. If one wishes to follow the successful route of the heavy-fermion theory from impurity to lattice once again, in the current intensive search of non-Fermi-liquid models to describe the normal state of high-T_c cuprates, only non-degenerate fixed points of an impurity model have the chance to be a successful starting point. Developments along this direction in the search of a promising impurity model have been reported recently [40]. It is worthwhile mentioning that a quantum critical system with a Fermi surface has a true coherent non-Fermi-liquid infrared fixed point in high dimensions. An example is an electron gas coupled to a transverse gauge field, where a self-consistent solution of the infrared fixed point has been derived [41]. It is interesting to note that this system is critical at $T = 0$ in 2D and 3D but without developing long-range order.

Acknowledgments

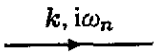
The author acknowledges useful conversations with D Cox, N Prokof'ev, P Stamp, A Tsvelick and C Varma, and especially his indebtedness to I Affleck, N Andrei and P Coleman whose patient explanation has been indispensable to the author's understanding of this subject. This work was supported in part by NSERC and CIAR of Canada.

Appendix A. Feynman rules

To be self-contained, we list the rules for constructing Feynman diagrams in the presence of an external magnetic field. These rules also define our convention.

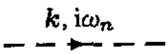
(i) For the contributions of nth order in the Kondo interaction, $\mathcal{O}(J^n)$, draw all topologically distinct diagrams with n vertices.

(ii) For each conduction electron propagator, draw a solid line:



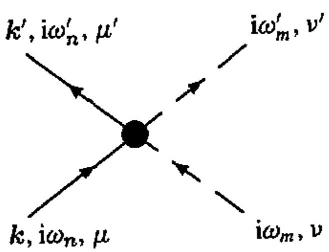
$$= G^{(0)}_{\mu\nu}(k, \mathrm{i}\omega_n) = \frac{-\delta_{\mu\nu}}{\mathrm{i}\omega_n - \epsilon_k - \mu h}. \tag{A1}$$

There is a summation over each momentum k of the internal conduction electron propagator. For each pseudofermion propagator, draw a broken line:



$$= \mathcal{G}^{(0)}_{\mu\nu}(\mathrm{i}\omega_n) = \frac{-\delta_{\mu\nu}}{\mathrm{i}\omega_n - \mathrm{i}\omega_0 - \mu\mu_S h}. \tag{A2}$$

(iii) Each vertex is associated with a factor:



$$= -\frac{J}{4\mathcal{N}\beta}\, \sigma_{\mu'\mu} \cdot \sigma_{\nu'\nu}\, \delta_{\omega_n+\omega_m, \omega'_n+\omega'_m}. \tag{A3}$$

(iv) Each independent internal frequency is summed over.

(v) Each conduction electron loop contributes a factor $-k$ and each pseudofermion loop contributes a factor -1.

(vi) For a diagram of order J^n, there is a numerical factor: (number of different connections)/n!. This combinatorial factor will be given explicitly in the figures of this paper for the diagrams we calculate.

Appendix B. Dynamical spin–spin correlation functions

Three dynamical correlation functions have been defined in sections 8 and 9. To order $\mathcal{O}(1) + \mathcal{O}(1/k)$, their diagrams are shown in figure 4. Each diagram contains two external vertices where the external spin operators reside. They are represented by two open ends at the left and right sides. The external vertex at each end is joined by two propagators. If the spin indices of these two propagators are σ_1 and σ_2, the corresponding vertex is associated with a vector of spin-$\frac{1}{2}$ matrices $s_{\sigma_2\sigma_1}$. The two vectors of matrices from the two external vertices at the two ends form a scalar product $s_{\sigma_2\sigma_1} \cdot s_{\sigma_4\sigma_3}$. In χ_e, only one of the two external vertices has the two joining conduction electron propagators differing in their momenta by q. At the other external vertex, the incoming and outgoing conduction electron momenta are the same.

The result for $\chi_f(\mathrm{i}\nu_n)$ is given by (55). The results for the other two are

$$\chi_{fe}(q, \mathrm{i}\nu_n) = -\delta_{n,0}\tfrac{3}{8}gk\beta\{\Pi_0(q)[1 - g^2k(\ln\beta D - \ln 2 + I_0)] + g\ln 2\Pi_1(q)\} + (1 - \delta_{n,0})\chi_{fe}(q, \mathrm{i}\nu_n) \tag{B1}$$

$$\chi_e(q, \mathrm{i}\nu_n) = -\delta_{n,0}\tfrac{1}{2}gk\chi_{fe}(q, \mathrm{i}\nu_n) + \delta_{n,0}\frac{3\ln 2}{16}g^3k^2\beta\Pi_0(q) - (1 - \delta_{n,0})\chi_{fe}(q, \mathrm{i}\nu_n) \tag{B2}$$

where Π_0 and Π_1 are defined in section 10, and for $\nu_n \neq 0$

$$\chi_{fe}(q, \mathrm{i}\nu_n) = \tfrac{3}{8}g^2k[gk\Pi_0(q)K(\mathrm{i}\nu_n) + 2K(q, \mathrm{i}\nu_n) - 2L(q, \mathrm{i}\nu_n)] \tag{B3}$$

$$K(q, \mathrm{i}\nu_n) = \frac{1}{\rho^2\mathcal{N}^2}\sum_{k,k'}\frac{1}{\epsilon_k - \epsilon_{k'+q}}\frac{f(\epsilon_k) - f(\epsilon_{k'})}{(\epsilon_k - \epsilon_{k'})^2 + \nu_n^2} \tag{B4}$$

$$L(q, \mathrm{i}\nu_n) = \frac{1}{\rho^2\mathcal{N}^2}\sum_{k,k'}\frac{1}{\epsilon_k - \epsilon_{k'}}\frac{f(\epsilon_{k'}) - f(\epsilon_{k'+q})}{(\epsilon_{k'} - \epsilon_{k'+q})^2 + \nu_n^2}. \tag{B5}$$

Note that $K(q = 0, \mathrm{i}\nu_n) = K(\mathrm{i}\nu_n)$ is a generalization of $K(\mathrm{i}\nu_n)$ defined by (56) to finite q. In the expressions for χ_{fe} and χ_e, we have dropped contributions of order $1/D$ to the $\nu_n = 0$ components comparing to the contributions of order β we kept.

References

[1] Muramatsu A and Guinea F 1986 *Phys. Rev. Lett.* **57** 2337
[2] Cox D L 1987 *Phys. Rev. Lett.* **59** 1240
[3] Seaman C L, Maple M B, Lee B W, Ghamaty S, Torikachvili M S, Kang J S, Liu L Z, Allen J W and Cox D L 1991 *Phys. Rev. Lett.* **67** 2882
[4] Andraka B and Tsvelick A M 1991 *Phys. Rev. Lett.* **67** 2886
[5] Ralph D C and Buhrman R A 1992 *Phys. Rev. Lett.* **69** 2118
[6] Emery V J and Kivelson S A 1993 *Physica* C **209** 597
[7] Cox D L 1988 *Physica* C **153–5** 1642
[8] Tsvelick A M 1990 *J. Phys.: Condens. Matter* **2** 2833
[9] Affleck I and Ludwig A W W 1991 *Nucl. Phys.* B **360** 641
[10] Affleck I and Ludwig A W W 1991 *Phys. Rev. Lett.* **67** 161
[11] Affleck I and Ludwig A W W 1991 *Phys. Rev. Lett.* **67** 3163
[12] Sacramento P D and Schlottman P 1991 *J. Phys.: Condens. Matter* **3** 9687
[13] Sacramento P D and Schlottman P 1991 *Phys. Rev.* B **43** 13294
[14] Emery V J and Kivelson S A 1992 *Phys. Rev.* B **46** 10812
[15] Gan J, Andrei N and Coleman P 1993 *Phys. Rev. Lett.* **70** 686

[16] Affleck I and Ludwig A W W 1993 *Phys. Rev.* B **48** 7297
[17] Cox D L and Ruckenstein A E 1993 *Phys. Rev. Lett.* **71** 1613
[18] Emery V J and Kivelson S A 1993 *Phys. Rev. Lett.* **71** 3701
[19] Nozières P 1974 *J. Low Temp. Phys.* **17** 31
[20] Lee P A, Rice T M, Serene J W, Sham L J and Wilkins J W 1986 *Comment. Condens. Matter Phys.* **12** 99
[21] Anderson P W 1992 *Science* **256** 1526
[22] Varma C M 1989 *Int. J. Mod. Phys.* **B 3** 2083
Varma C M, Littlewood P B, Schmitt-Rink S, Abrahams E and Ruckenstein A E 1989 *Phys. Rev. Lett.* **63** 1996
[23] Nozières P and Blandin A 1980 *J. Physique* **41** 193
[24] Cragg D M, Lloyd P and Nozières P 1980 *J. Phys. C: Solid State Phys.* **13** 803
[25] Andrei N and Destri C 1984 *Phys. Rev. Lett.* **52** 364
[26] Tsvelick A M and Wiegmann P B 1984 *Z. Phys.* B **54** 201
[27] Nozières P 1985 *Ann. Phys., Paris* **10** 19
[28] Boyce J and Slichter C 1974 *Phys. Rev. Lett.* **32** 61
[29] Anderson P W 1984 *Moment Formation in Solids* ed W J L Buyers (New York: Plenum) p 313
[30] Varma C M 1987 *Fermi Surface Effects* ed J Kondo and A Yoshimori (Berlin: Springer) p 117
[31] Popov V N and Fedotov S A 1988 *Sov. Phys.–JETP* **67** 535
[32] Mahan G D 1990 *Many-Particle Physics* 2nd edn (New York: Plenum)
[33] Abrikosov A A 1965 *Physics* **2** 5
[34] Collins J 1984 *Renormalization* (Cambridge: Cambridge University Press) p 193
[35] Kondo J 1968 *Prog. Theor. Phys.* **40** 683
[36] Ashcroft N W and Mermin N D 1976 *Solid State Physics* (New York: Saunders) pp 47, 663
[37] Lowenstein J H 1984 *Phys. Rev.* B **29** 4120. This conclusion can be proved simply by relabelling the conduction electron momenta in the Hamiltonian.
[38] Ruderman M A and Kittel C 1954 *Phys. Rev.* **96** 99
Kasuya T 1956 *Prog. Theor. Phys.* **16** 45
Yosida K 1957 *Phys. Rev.* **106** 893
[39] Ingersent K, Jones B A and Wilkins J W 1992 *Phys. Rev. Lett.* **69** 2594
[40] Giamarchi T, Varma C M, Ruckenstein A E and Nozières P 1993 *Phys. Rev. Lett.* **70** 3967
[41] Gan J and Wong E 1993 *Phys. Rev. Lett.* **71** 4226